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Relativistically Corrected Masses of Ground-State Baryons in Hyperspherical Harmonic Quark Model

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The masses of the ground-state light baryons are calculated in the quark model. The unperturbed wave functions correspond to a hyperspherical harmonic confinement. The perturbation includes a short-range potential and all the relevant relativistic corrections of the order of v^2/c^2. Results are compared with the experimental values and found to be in good agreement. This may be a test not of the hyperspherical harmonic model so much as of the feasibility of a simple (but consistent) relativistically corrected fit of the light baryon masses.

1. INTRODUCTION

In a previous paper (Raspini, 1988) we proposed a hyperspherical harmonic model of the 18 ground-state light baryons. The employed non-relativistic Hamiltonian

$$H_{(0)} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \frac{K}{2} \sum_{i \neq j} \frac{m_i m_j}{2M^2} (\mathbf{r}_i - \mathbf{r}_j)^2$$
$$i, j = 1, 2, 3; \qquad M = \sum_i m_i \tag{1}$$

includes the kinetic energy and a long-range (confining) hypercentral harmonic potential (Fabre De La Ripelle, 1984), with elastic constant of fixed value[3] K. As usual, $\mathbf{p}_i$ are the canonical momenta, $\mathbf{r}_i$ the positions, and m_i the effective (constituent) masses of the three quarks; units will be such

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[3]A straightforward and "natural" modification is that of a mass-dependent elastic constant: $K = K_0 M/\alpha(m_1, m_2, m_3)$, where K_0 has a fixed positive value and $\alpha > 0$ is chosen to be totally symmetric ($\alpha = M$ reproduces the fixed-K case).

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that $\hbar = c = 1$. The wave functions are as follows, in the center of mass (cm) frame of reference (Raspini, 1988):

$$|\Psi_{\text{cm}}(gb, S)\rangle = |Bn = +1;\ W\rangle \times \left\{ \left(\frac{K m_1 m_2 m_3}{\pi^2 M^2 D^2} \right)^{3/4} \exp\left[-\frac{1}{2} (MK)^{1/2} \xi^2 \right] \right\} |\mathcal{U}(gb, S)\rangle \tag{2}$$

where (i) gb stands for the name of the baryon (e.g., Δ^{++}, Δ^{+}, etc.); (ii) S indicates the z-spin eigenstate of the baryon ($S = -3/2, -1/2, +1/2, +3/2$ for the decuplet, $S = -1/2, +1/2$ for the octet); (iii) the ket $|Bn = +1;\ W\rangle$ represents the (normalized) color baryon number wave function of a colorless (W) baryonic state of baryon number $Bn = +1$ (for antibaryons, $gb \to \overline{gb}$ by means of $Bn \to -1$); (iv) $\xi \geq 0$ is the hyperradial variable

$$\xi^2 = \sum_{i \neq j} \frac{m_i m_j}{2M^2} (\mathbf{r}_i - \mathbf{r}_j)^2 \tag{3}$$

and $D^3 \to \infty$ stands for the volume of the three-dimensional space, according to the convention

$$\int d\mathbf{R} = D^3 \tag{4}$$

where $\mathbf{R}$ is the cm position of the baryon. The space eigenfunction (in curly brackets) is then normalized with respect to the nine-dimensional volume

$$dv = \prod_i d\mathbf{r}_i \tag{5}$$

and (v) the ket $|\mathcal{U}(gb, S)\rangle = |\mathcal{U}(\overline{gb}, S)\rangle$ denotes the flavor spin wave function, which has the general form

$$|\mathcal{U}(gb, S)\rangle = \sum_F C_F(gb, S) |fl', s'\rangle_1 |fl'', s''\rangle_2 |fl''', s'''\rangle_3$$
$$F = (fl', s', fl'', s'', fl''', s''') \tag{6}$$

if fl are the quark flavors [up (up), down (dw), strange (st)] and s the quark z-spin eigenvalues $(-1/2, +1/2)$. The $C_F(gb, S)$ coefficients are chosen according to well-known rules, including orthonormalization and symmetry. For example (Perkins, 1982)

$$|\mathcal{U}(\Delta^{++}, +3/2)\rangle = |up, +1/2\rangle_1 |up, +1/2\rangle_2 |up, +1/2\rangle_3 \tag{7}$$

that is,

$$C_F(\Delta^{++}, +3/2) = \delta(fl', up)\delta(s', +1/2)\delta(fl'', up)\delta(s'', +1/2) \times \delta(fl''', up)\delta(s''', +1/2) \tag{8}$$

where $\delta(,)$ is the Kronecker delta; note in particular the relations

$${}_i\langle fl | \widetilde{fl}\rangle_i = \delta(fl, \widetilde{fl}) \tag{9}$$

$${}_i\langle s | \tilde{s}\rangle_i = \delta(s, \tilde{s}) \tag{10}$$

Furthermore, the space wave function formally operates on the SU(6) wave function in the following way (Raspini, 1988):

$$m_i|fl\rangle_i = m(fl)|fl\rangle_i \tag{11}$$

where $m(fl)$ is the fl-eigenvalue of the m_i mass operator [that is, the mass of the quark in $|\cdot\rangle_i$: see also Messiah (1966), Chapter XIV, §14]. For the purpose of clarity, we recall here that each baryon gb has a "permutational" flavor contents $(qa, qb, qc)|_{gb}$, with $qa, qb, qc \in \{up, dw, st\}$. That is, the non-vanishing $C_F(gb, S)$ coefficients are among those for which (fl', fl'', fl''') is a permutation of $(qa, qb, qc)|_{gb}$ (Perkins, 1982). Hence, if $\Theta(m_1, m_2, m_3)$ is totally symmetric, $\Theta|\mathcal{U}(gb, S)\rangle = \theta(gb)|\mathcal{U}(gb, S)\rangle$, where θ is the obvious eigenvalue. (In particular, this applies to the mass M.)

In the present paper, the described model will be utilized for the calculation of the masses of the 18 baryons. This will be done considering $H_{(0)}$ as the unperturbed Hamiltonian, and adding to it the following perturbations: (a) the v^2/c^2 relativistic correction to the kinetic energy; (b) the v^2/c^2 relativistic corrections to the confining potential; and (c) a short-range potential, including v^2/c^2 relativistic terms.

2. CORRECTIONS TO THE KINETIC ENERGY AND CONFINING POTENTIAL

The expression of the v^2/c^2 correction to the kinetic energy has the standard form (Grotch and Sebastian, 1982)

$$H_{(1)} = \sum_i -\frac{\mathbf{p}_i^4}{8m_i^3} \tag{12}$$

For the calculation of the v^2/c^2 corrections to the confining potential, we apply to each pair of quarks the general prescriptions outlined by Grotch and Sebastian (1982), based on the Barker-Glover reduction of the two-body Dirac Hamiltonian. If the confining potential of equation (1) is considered to be a scalar, we obtain, after some simple algebra,

$$H_{(2)} = \sum_{i\neq j} -\frac{Km_j}{4m_iM^2}\left(\mathbf{p}_i r_{ij}^2 \cdot \mathbf{p}_i - \frac{3}{2}\right) + \sum_{i\neq j} -\frac{Km_j}{2m_iM^2}\mathbf{S}_i \cdot (\mathbf{r}_{ij} \times \mathbf{p}_i) \tag{13}$$

where

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j; \qquad r_{ij} = |\mathbf{r}_{ij}| \tag{14}$$

and $\mathbf{S}_i$ are the quark spin operators. On the other hand, if the confining potential of equation (1) has a vector nature, the set of corrections is different:

$$H_{(3)}=\sum_{i\neq j}\frac{3K}{16M^2}\frac{(m_i+m_j)^2+m_im_j}{m_im_j}$$
$$+\sum_{i\neq j}-\frac{K}{8M^2}[\mathbf{p}_i r_{ij}^2\cdot\mathbf{p}_j-2(\mathbf{p}_i\cdot\mathbf{r}_{ij})(\mathbf{r}_{ij}\cdot\mathbf{p}_j)]$$
$$+\sum_{i\neq j}\frac{Km_j}{M^2}\mathbf{S}_{ij}\cdot(\mathbf{r}_{ij}\times\mathbf{p}_i)+\sum_{i\neq j}\frac{K}{M^2}\mathbf{S}_i\cdot\mathbf{S}_j \tag{15}$$

with

$$\mathbf{S}_{ij}=\frac{\mathbf{S}_i}{2m_i}+\frac{\mathbf{S}_j}{m_j} \tag{16}$$

The latter case will be here discarded, since the hypothesis of a scalar confinement is more widely accepted: see, for example, Gupta *et al.* (1982). For a generalization, note the possibility of a partly scalar and partly vector confinement (Grotch *et al.*, 1984).

3. SHORT-RANGE POTENTIAL

The nonrelativistic short-range potential will be taken to have the following form (Sebastian, 1982):

$$H_{(4)}=\sum_{i\neq j}\frac{K_{ij}}{2r_{ij}} \tag{17}$$

with (Rosner, 1981)

$$K_{ij}=e_ie_j-\tfrac{2}{3}\gamma_{ij} \tag{18}$$

where e_i are the quark charges and $\gamma_{ij}=\gamma_{ji}$ plays the role of the (effective) strong coupling constant between the quark labeled i and the quark labeled j. Since the short-range potential has a vector nature (Sebastian, 1982), the v^2/c^2 corrections turn out to be (Grotch and Sebastian, 1982; see also Sebastian, 1982; Raspini, 1985)

$$H_{(5)}=\sum_{i\neq j}-\frac{K_{ij}}{4m_im_j}\left(\mathbf{p}_i\frac{1}{r_{ij}}\cdot\mathbf{p}_j\right)$$
$$+\sum_{i\neq j}-\frac{K_{ij}}{4m_im_j}\left(\mathbf{p}_i\cdot\mathbf{r}_{ij}\frac{1}{r_{ij}^3}\mathbf{r}_{ij}\cdot\mathbf{p}_j\right)+\sum_{i\neq j}-\frac{K_{ij}}{m_i}\mathbf{S}_{ij}\cdot\left(\frac{\mathbf{r}_{ij}}{r_{ij}^3}\times\mathbf{p}_i\right)$$
$$+\sum_{i\neq j}-\frac{K_{ij}\pi}{2}\left[\left(\frac{1}{2m_i^2}+\frac{1}{2m_j^2}\right)+\frac{8}{3m_im_j}\mathbf{S}_i\cdot\mathbf{S}_j\right]\delta(\mathbf{r}_{ij})$$
$$+\sum_{i\neq j}\frac{K_{ij}}{2m_im_j}\frac{1}{r_{ij}^3}\left[\mathbf{S}_i\cdot\mathbf{S}_j-\frac{3(\mathbf{S}_i\cdot\mathbf{r}_{ij})(\mathbf{S}_j\cdot\mathbf{r}_{ij})}{r_{ij}^2}\right] \tag{19}$$

We here take

$$\gamma_{ij}=\gamma\left[1+\frac{9\gamma}{4\pi}\ln\left(\frac{3m_i+3m_j}{2M}\right)\right]^{-1} \tag{20}$$

with $\gamma=1.50$. This is based on the formula

$$\gamma_{ij}=\gamma\left[1+\frac{(33-2\nu)\gamma}{12\pi}\ln\left(\frac{m_{ij}}{Z}\right)\right]^{-1} \tag{21}$$

where ν stands for the number of relevant different flavors ($\nu=3$, in our case), $m_{ij}=m_{ji}$ is a mass scale for the ij quark pair, and Z indicates the reference mass value. [See equations (3.1), (3.2) of Gupta *et al.* (1982).] Assuming an intrinsic mass scale Z_0, identical for all the ground-state light baryons, a possible estimate is as follows:

$$m_{ij}\simeq\frac{m_i+m_j}{M}Z_0 \tag{22}$$

and equation (20) finally results, choosing $Z=2Z_0/3$ (so that γ corresponds to quarks in Δ^{++},Δ^-, or Ω). In order to decrease the number of free parameters, γ was preselected to be 1.50, which is a rough average of values adopted for similar mass fit purposes in different models (Heller, 1982). As for the quark charges, we observe that (Raspini, 1988)

$$\begin{aligned}&|Bn=\pm1\rangle=|bn=Bn/3\rangle_1|bn=Bn/3\rangle_2|bn=Bn/3\rangle_3\\&{}_i\langle bn|b\tilde{n}\rangle_i=\delta(bn,b\tilde{n})\end{aligned} \tag{23}$$

where bn indicates the quark baryon number. Then

$$e_i|fl,bn\rangle_i=e(fl,bn)|fl,bn\rangle_i \tag{24}$$

with $e(up,\pm1/3)=\pm2/3$ and $e(dw,\pm1/3)=e(st,\pm1/3)=\mp1/3$ (in units of the proton's charge).

4. CM VARIABLES

The various Hamiltonian operators here introduced [equations (1), (12), (13), (15), (17), and (19)] can all be expressed in terms of relative variables $\boldsymbol{\pi}_i$, $\boldsymbol{\rho}_i$, total momentum $\mathbf{P}=\sum_i\mathbf{p}_i$, and cm position $\mathbf{R}=(\sum_i m_i\mathbf{r}_i)/M$, according to the equations (Raspini, 1985, 1988)

$$\mathbf{p}_i=\boldsymbol{\pi}_i+\frac{m_i}{M}\mathbf{P} \tag{25}$$

$$\mathbf{r}_i=\boldsymbol{\rho}_i+\mathbf{R} \tag{26}$$

with constraints

$$\sum_i \boldsymbol{\pi}_i = \mathbf{0}; \qquad \sum_i m_i \boldsymbol{\rho}_i = \mathbf{0} \tag{27}$$

The only nonvanishing commutators are

$$[\rho_i^a, \pi_j^b] = \mathrm{i}\delta^{ab}(\delta_{ij} - m_j/M) \tag{28}$$

$$[R^a, P^b] = \mathrm{i}\delta^{ab} \tag{29}$$

where superscripts indicate Cartesian components.

If we define the coordinates

$$\boldsymbol{\eta}_1 = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_3 = \mathbf{r}_1 - \mathbf{r}_3 \tag{30}$$

$$\boldsymbol{\eta}_2 = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_3 = \mathbf{r}_2 - \mathbf{r}_3 \tag{31}$$

we can use $\mathbf{R}$, $\boldsymbol{\eta}_1$, and $\boldsymbol{\eta}_2$ to span the whole nine-dimensional space,

$$\mathbf{r}_1 = \mathbf{R} + \frac{1}{M}[(m_2 + m_3)\boldsymbol{\eta}_1 - m_2\boldsymbol{\eta}_2] \tag{32}$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{1}{M}[m_1\boldsymbol{\eta}_1 - (m_1 + m_3)\boldsymbol{\eta}_2] \tag{33}$$

$$\mathbf{r}_3 = \mathbf{R} - \frac{1}{M}(m_1\boldsymbol{\eta}_1 + m_2\boldsymbol{\eta}_2) \tag{34}$$

Furthermore (Raspini, 1988),

$$P^a = -\mathrm{i}\frac{\partial}{\partial R^a} \tag{35}$$

$$\pi_1^a = -\mathrm{i}\frac{\partial}{\partial \eta_1^a}; \qquad \pi_2^a = -\mathrm{i}\frac{\partial}{\partial \eta_2^a} \tag{36}$$

$$\pi_3^a = \mathrm{i}\frac{\partial}{\partial \eta_1^a} + \mathrm{i}\frac{\partial}{\partial \eta_2^a} \tag{37}$$

For each of the Hamiltonian operators $H_{(\beta)}(\mathbf{p}_i, \mathbf{r}_i, \mathbf{S}_i)$, $\beta = 0, \ldots, 5$, the replacements (25) and (26) may be employed to show that

$$H_{(\beta)}|\Psi_{\mathrm{cm}}\rangle = h_{(\beta)}|\Psi_{\mathrm{cm}}\rangle \tag{38}$$

where

$$h_{(\beta)} = H_{(\beta)}(\boldsymbol{\pi}_i, \boldsymbol{\rho}_i, \mathbf{S}_i) \tag{39}$$

and $|\Psi_{\mathrm{cm}}\rangle$ is any cm state:

$$\mathbf{P}|\Psi_{\mathrm{cm}}\rangle = \mathbf{0} \tag{40}$$

To operate on the wave functions (2), we can then use the $h_{(\beta)}$, in which the $\boldsymbol{\rho}_i$ variables are expressed in terms of $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ [equations (26) and (32)-(34)] and the $\boldsymbol{\pi}_i$ variables by means of equations (36) and (37). The volume element (5) and the hyperradius (3) are as follows (Raspini, 1988):

$$dv = d\mathbf{R}\, d\boldsymbol{\eta}_1\, d\boldsymbol{\eta}_2$$

$$\xi^2 = \frac{1}{M^2}[m_1(m_2+m_3)\boldsymbol{\eta}_1^2 + m_2(m_1+m_3)\boldsymbol{\eta}_2^2 - 2m_1 m_2 \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2] \qquad (41)$$

5. MASSES OF THE BARYONS

It is here convenient to define

$$h = M + \sum_{\beta \neq 3} h_{(\beta)} \qquad \text{(scalar confinement)} \qquad (42)$$

Also, $\Phi = \Phi^*$ will indicate the space wave function of equation (2). In first-order perturbation theory, the masses of the baryons may be calculated as follows:

$$\mathcal{M}(gb) = \langle \Psi_{\rm cm}(gb, S)|h|\Psi_{\rm cm}(gb, S)\rangle \qquad (43)$$

which gives, after some simple algebra

$$\mathcal{M}(gb) = \sum_{F,\tilde{F}} C_F^*(gb, S) C_{\tilde{F}}(gb, S)\delta(fl', \widetilde{fl'})\delta(fl'', \widetilde{fl''})\delta(fl''', \widetilde{fl'''})$$
$$\times \left\{ \int dv\, ({}_3\langle s'''|\,{}_2\langle s''|\,{}_1\langle s'|\Phi h \Phi|\tilde{s}'\rangle_1 |\tilde{s}''\rangle_2 |\tilde{s}'''\rangle_3) \right\} \qquad (44)$$

where the spin space matrix element, in curly brackets, is to be evaluated with $m_1 = m(fl')$, $e_1 = e(fl', +1/3)$, $m_2 = m(fl'')$, etc., according to equations (11), (24). [Obviously, $Bn \to -Bn$ changes the charges but does not change results, that is, $\mathcal{M}(gb) = \mathcal{M}(\overline{gb})$.] The computation of the matrix elements needed in equation (44) is tedious but straightforward: a numerical fit then gives the results of Table I. The small mass splitting between *up* and *dw* has been chosen to reproduce, roughly, the proton–neutron mass splitting.[4] The table, however, does not show the different states of charge for each hadron: the displayed mass values are averaged over these states. See also Rosner (1981).

While doing the fit, we kept $m(up)$ and ω_0 "reasonably" close to some reference values. For $m(up)$, the reference value was $(\Delta + N)/6$, which is a crude, but sound, estimate (Rosner, 1981). (Hadron symbols here indicate experimental masses.) As for ω_0, a rough reference value is obtainable from

[4]If $m(up) = m(dw)$, the proton comes out slightly heavier than the neutron, due to the internal electromagnetic interaction of the constituent quarks [see equation (18)].

Table I. Masses of the Baryons (MeV)

	Experimental	Calculated
N	939	939
Λ	1116	1119
Σ	1193	1158
Δ	1232	1208
Ξ	1318	1316
Σ^*	1384	1379
Ξ^*	1533	1544
Ω	1672	1699

$m(up) = 558$ MeV $m(dw) = 560$ MeV
$m(st) = 721$ MeV
$\omega_0 \equiv [K/3m(up)]^{1/2} = 398$ MeV

the experimental baryon spectrum, considering the mass spacings between our ground-state Δ and the known excited states of the same type (Rosner, 1981). We recall that the cm energy levels of the confining Hamiltonian (1) are as follows (Raspini, 1988):

$$\varepsilon_n = (n+3)\left(\frac{K}{M}\right)^{1/2}, \qquad n = 0, 1, \ldots \tag{45}$$

so that

$$(\varepsilon_n - \varepsilon_0)|_{\Delta^{++}\,\text{type}} = n\omega_0 \tag{46}$$

Some doubts obviously exist about the validity of the perturbation treatment, especially of the nonrelativistic short-range potential: a possible improvement of the method would have to take into account the modifications to the wave functions due to the presence of (17). Overall, the mass fit is satisfactory, with an average error of the order of 1%. This is certainly interesting, given the simplicity and consistency of the model. In fact, most *ad hoc* fits are only better by a factor of two in the average percent error (e.g., Rosner, 1981).

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